Lecture 7
Divide and conquer (cont.), master theorem, integer multiplication, maxima set

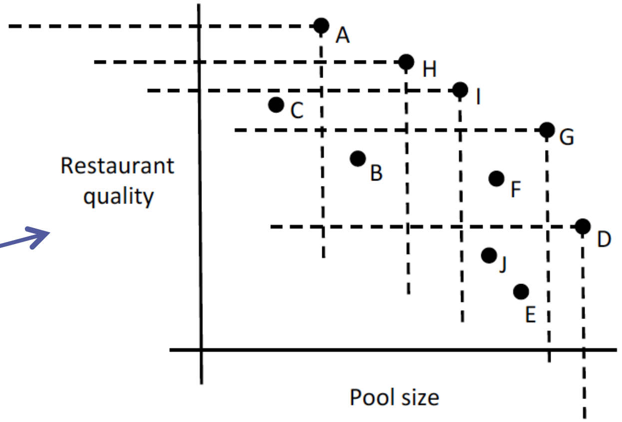
CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Application: Maxima Sets

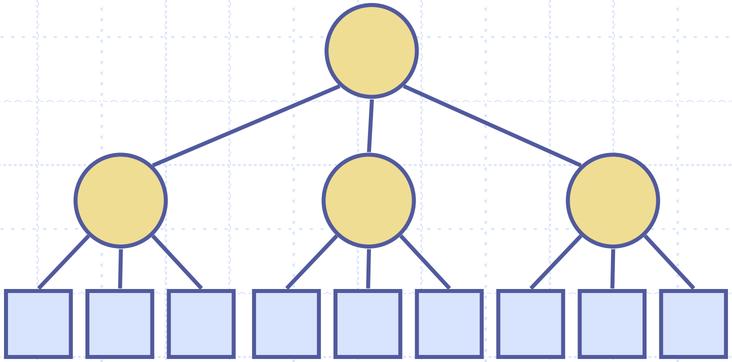
- We can visualize the various trade-offs for optimizing two-dimensional data, such as points representing hotels according to their pool size and restaurant quality, by plotting each as a two-dimensional point, (x, y), where x is the pool size and y is the restaurant quality score.
- We say that such a point is a **maximum point** in a set if there is no other point, (x′, y′), in that set such that x ≤ x′ and y ≤ y′.
- The maximum points are the best potential choices based on these two dimensions and finding all of them is the **maxima set** problem.

We can efficiently find all the maxima points by divide-and-conquer. Here the set is {A,H,I,G,D}.



Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data S in two or more disjoint subsets S_1, S_2, …
  - Conquer: solve the subproblems recursively
  - Combine: combine the solutions for S_1, S_2, …, into a solution for S
- The base case for the recursion are subproblems of constant size
- Analysis can be done using recurrence equations



Merge-Sort Review

- Merge-sort on an input sequence S with n elements consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - Conquer: recursively sort S_1 and S_2
  - Combine: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S)
  Input sequence S with n elements
  Output sequence S sorted according to C
  if $S.size() > 1$
    $(S_1, S_2) \leftarrow partition(S, n/2)$
    $mergeSort(S_1)$
    $mergeSort(S_2)$
    $S \leftarrow merge(S_1, S_2)$

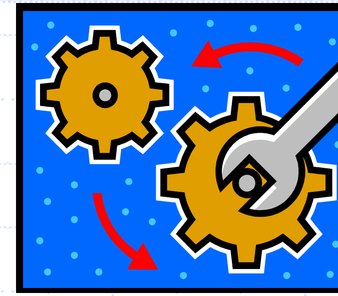
Recurrence Equation Analysis

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes at most bn steps, for some constant b.
- Likewise, the basis case ($n < 2$) will take at b most steps.
- Therefore, if we let $T(n)$ denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

- We can therefore analyze the running time of merge-sort by finding a closed form solution to the above equation.
  - That is, a solution that has $T(n)$ only on the left-hand side.

Iterative Substitution



- In the iterative substitution, or “plug-and-chug,” technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern:

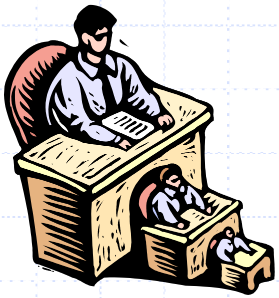
$$
\begin{aligned}
T(n) &= 2T(n/2) + bn \\
&= 2(2T(n/2^2)) + b(n/2)) + bn \\
&= 2^2T(n/2^2) + 2bn \\
&= 2^3T(n/2^3) + 3bn \\
&= 2^4T(n/2^4) + 4bn \\
&= \ldots \\
&= 2^iT(n/2^i) + ibn
\end{aligned}
$$

- Note that base, T(n)=b, case occurs when $2^i=n$. That is, i = log n.
- So,

$$T(n) = bn + bn\log n$$

- Thus, T(n) is O(n log n).

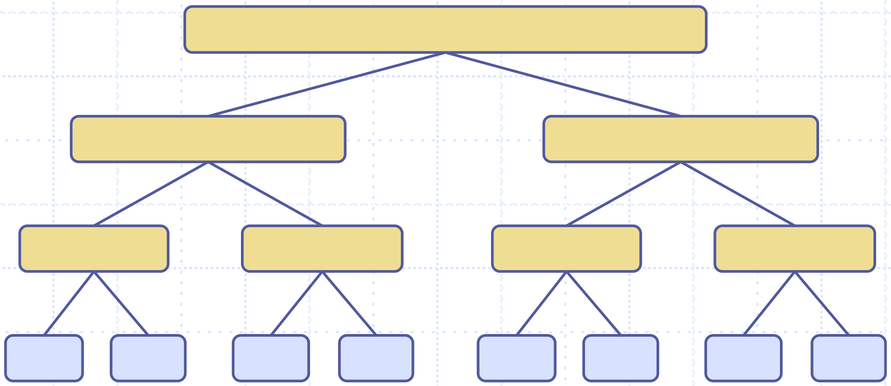
The Recursion Tree



- Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

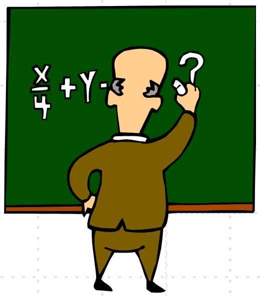
depth	T's	size	time
0	1	n	bn
1	2	$n/2$	bn
i	2^i	$n/2^i$	bn
…	…	…	…



Total time = $bn + bn \log n$

(last level plus all previous levels)

Guess-and-Test Method



- In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

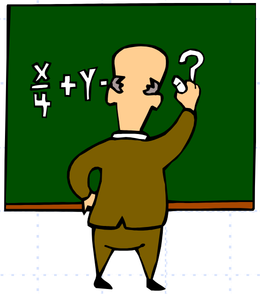
$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- Guess: T(n) < cn log n.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &= 2(c(n/2)\log(n/2)) + bn\log n \\ &= cn(\log n - \log 2) + bn\log n \\ &= cn\log n - cn + bn\log n \end{aligned}$$

- Wrong: we cannot make this last line be less than cn log n

Guess-and-Test Method, (cont.)



- Recall the recurrence equation:

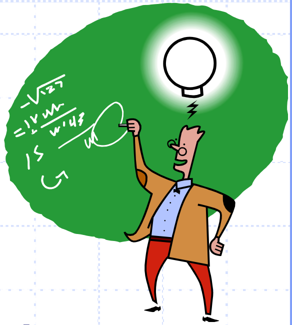
$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- Guess #2: $T(n) < cn \log^2 n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &= 2(c(n/2)\log^2(n/2)) + bn\log n \\ &= cn(\log n - \log 2)^2 + bn\log n \\ &= cn\log^2 n - 2cn\log n + cn + bn\log n \\ &\leq cn\log^2 n \end{aligned}$$

  - if c > b.
- So, T(n) is $O(n \log^2 n)$.
- In general, to use this method, you need to have a good guess and you need to be good at induction proofs.

Master Method



- Many divide-and-conquer recurrence equations have the form:

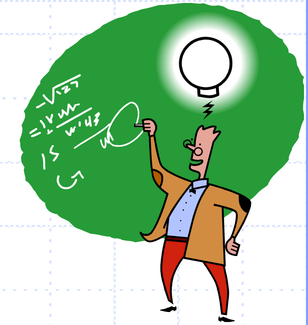
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

- The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Master Method, Example 1



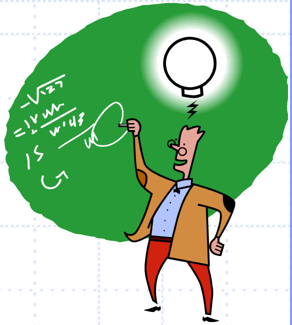
- The form: $$T(n)=\begin{cases} c & \text{if } n<d \\ aT(n/b)+f(n) & \text{if } n\ge d \end{cases}$$
- The Master Theorem:

  1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b)\le \delta f(n)$ for some $\delta<1$.

- Example: $$T(n)=4T(n/2)+n$$

Solution: $\log_b a=2$, so case 1 says $T(n)$ is $O(n^2)$.

Master Method, Example 2



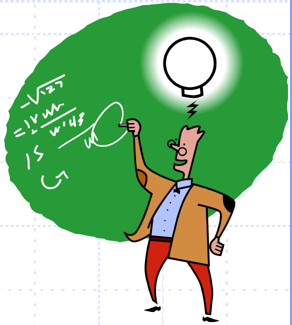
- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

     provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.
- Example:

$$T(n) = 2T(n/2) + n\log n$$

Solution: $\log_b a=1$, so case 2 says $T(n)$ is $O(n \log^2 n)$.

Master Method, Example 3



- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$
- The Master Theorem:

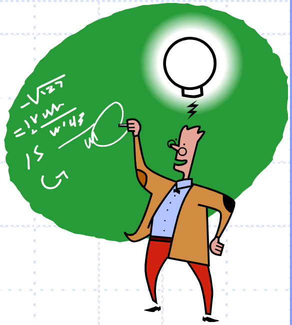
  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

     provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.
- Example:

$$T(n) = T(n/3) + n \log n$$

Solution: $\log_b a = 0$, so case 3 says T(n) is O(n log n).

Master Method, Example 4

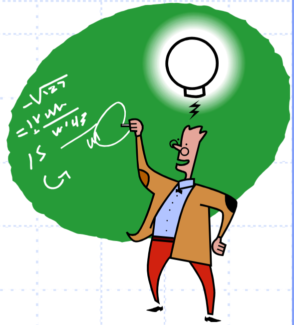


- The form: $$T(n)=\begin{cases} c & \text{if } n<d \\ aT(n/b)+f(n) & \text{if } n\ge d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
     provided $af(n/b)\le \delta f(n)$ for some $\delta<1$.
- Example:

$$T(n)=8T(n/2)+n^2$$

Solution: $\log_b a=3$, so case 1 says T(n) is $O(n^3)$.

Master Method, Example 5



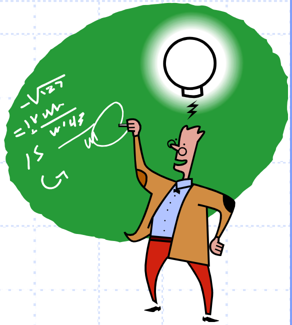
- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

     provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.
- Example:

$$T(n) = 9T(n/3) + n^3$$

Solution: $\log_b a=2$, so case 3 says T(n) is $O(n^3)$.

Master Method, Example 6

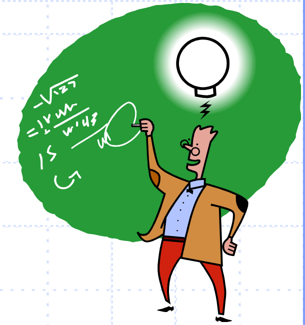


- The form: $$T(n)=\begin{cases} c & \text{if } n<d \\ aT(n/b)+f(n) & \text{if } n\ge d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
     provided $af(n/b)\le \delta f(n)$ for some $\delta<1$.
- Example:

$$T(n)=T(n/2)+1$$ (binary search)

Solution: $\log_b a=0$, so case 2 says T(n) is O(log n).

Master Method, Example 7



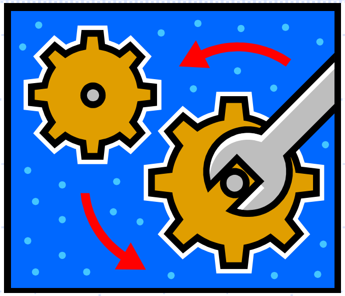
- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
- The Master Theorem:

  1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
  provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

- Example:

$$T(n) = 2T(n/2) + \log n$$ (heap construction)

Solution: $\log_b a=1$, so case 1 says T(n) is O(n).



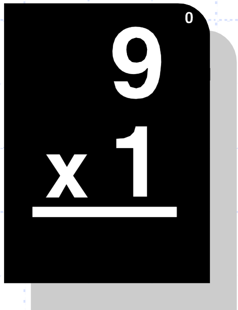
Sketch of Proof of the Master Theorem

- Using iterative substitution, let us see if we can find a pattern:

$$
\begin{aligned}
T(n) &= aT(n/b) + f(n) \\
&= a(aT(n/b^2)) + f(n/b)) + bn \\
&= a^2T(n/b^2) + af(n/b) + f(n) \\
&= a^3T(n/b^3) + a^2f(n/b^2) + af(n/b) + f(n) \\
&= \dots \\
&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\
&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)
\end{aligned}
$$

- We then distinguish the three cases as
  - The first term is dominant
  - Each part of the summation is equally dominant
  - The summation is a geometric series

Integer Multiplication



- Algorithm: Multiply two n-bit integers I and J.
  - Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

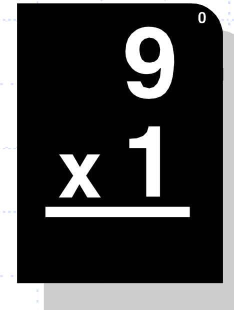
$$J = J_h 2^{n/2} + J_l$$

  - We can then define I*J by multiplying the parts and adding:

$$\begin{aligned} I * J &= (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l) \\ &= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \end{aligned}$$

  - So, $T(n) = 4T(n/2) + n$, which implies $T(n)$ is $O(n^2)$.
  - But that is no better than the algorithm we learned in grade school.

An Improved Integer Multiplication Algorithm



- Algorithm: Multiply two n-bit integers I and J.
  - Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

  - Observe that there is a different way to multiply parts:

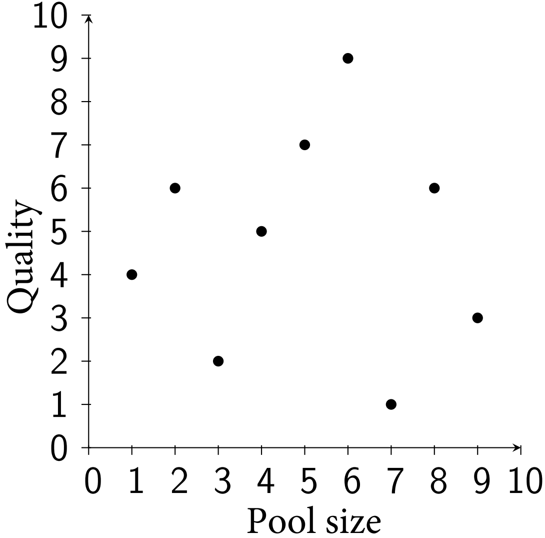
$$\begin{aligned} I * J &= I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l \end{aligned}$$

  - So, $T(n) = 3T(n/2) + n$, which implies $T(n)$ is $O(n^{\log_2 3})$, by the Master Theorem.
  - Thus, $T(n)$ is $O(n^{1.585})$.

Maxima Set Problem Statement

- We have a database of hotels.
- Each hotel has:
  - a pool size (x-coordinate)
  - quality of restaurant (y-coordinate)
  - Assume all coordinates distinct
- Want hotel with largest pool and best restaurant
  Might not be a unique hotel.
  - One might have largest pool, other best restaurant.
  - Return **the set** that aren't wrong.
    - Any where no other hotel has both larger pool and better restuarant.

Maxima Set Example



Minima Set Brute Force

Sort hotels along any dimension
for $i = 1 \rightarrow n - 1$ **do**
  for $j = i + 1 \rightarrow n$ **do**
    if A_i has larger pool and better food than A_j
      Remove A_j
return All hotels that we did not remove

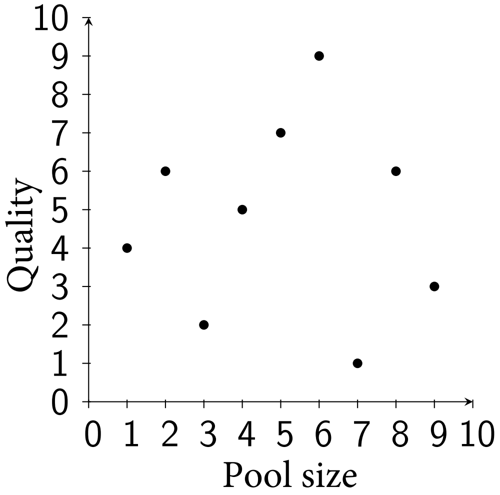
- This is $O(n^2)$.

Beginning Divide and Conquer

```
MaximaSet(S)
  if n ≤ 1 then
    return S
  p ← median point in S by x-coordinate
  L ← points less than p
  G ← points greater than or equal to p
  M1 ← MaximaSet(L)
  M2 ← MaximaSet(G)
```

- return $M_1 \cup M_2$?

Example revisited



- From $M_1 \cup M_2$, which point(s) belong for sure?

Finding a correct recombine

MaximaSet(S)
  if $n \leq 1$ **then**
    return S
  $p \leftarrow$ median point in S by x-coordinate
  $L \leftarrow$ points less than p
  $G \leftarrow$ points greater than **or equal to** p
  $M_1 \leftarrow$ MaximaSet(L)
  $M_2 \leftarrow$ MaximaSet(G)

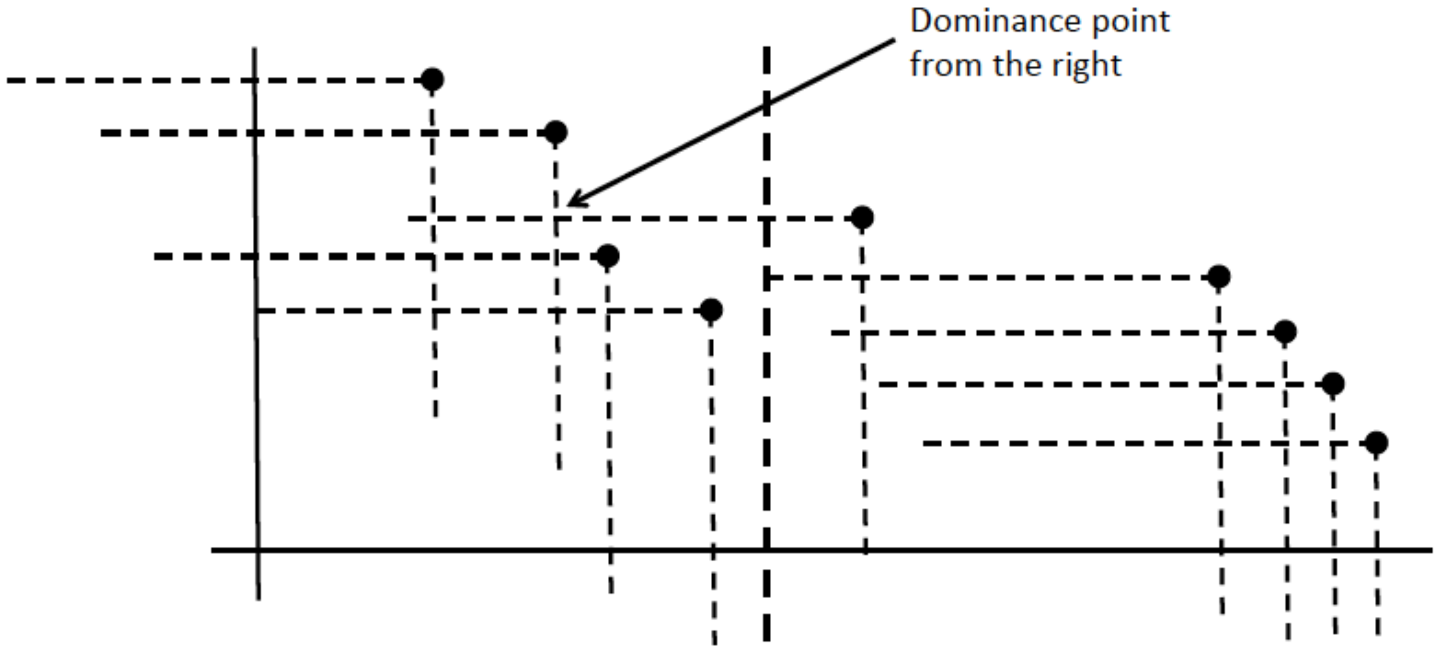
- return $M_1 \cup M_2$?
- How do I recombine correctly?

Improved Recombine

$M_1 \leftarrow$ `MaximaSet(`L`)`
$M_2 \leftarrow$ `MaximaSet(`G`)`
for each $a \in M_1$ **do**
  for each $b \in M_2$ **do**
    if a better than b **then**
      remove b from M_2

- How can we improve the "recombine" step?
- What is the resulting running time?

Example for the Combine Step



Analysis

- In either case, the rest of the non-recursive steps can be performed in O(n) time, so this implies that, ignoring floor and ceiling functions (as allowed by the analysis of Exercise C-11.5), the running time for the divide-and-conquer maxima-set algorithm can be specified as follows (where b is a constant):

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

- Thus, according to the Master Theorem, this algorithm runs in O(n log n) time.